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Bipartite Consensus of Discrete-Time Double-Integrator Multi-Agent Systems with Measurement Noise*

MA Cuiqin · ZHAO Weiwei · ZHAO Yunbo

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Abstract The effects of measurement noise are investigated in the context of bipartite consensus of multi-agent systems. In the system setting, discrete-time double-integrator dynamics are assumed for the agent, and measurement noise is present for the agent receiving the state information from its neighbors. Time-varying stochastic bipartite consensus protocols are designed in order to lessen the harmful effects of the noise. Consequently, the state transition matrix of the closed-loop system is analyzed, and sufficient and necessary conditions for the proposed protocol to be a mean square bipartite consensus protocol are given with the help of linear transformation and algebraic graph theory. It is proven that the signed digraph to be structurally balanced and having a spanning tree are the weakest communication assumptions for ensuring bipartite consensus. In particular, the proposed protocol is a mean square bipartite average consensus one if the signed digraph is also weight balanced.

Keywords Bipartite consensus, discrete-time, double-integrator, measurement noise, multi-agent systems.

1 Introduction

Consensus is a fundamental problem for multi-agent systems (MASs), attracting much attention from the research community[1–6]. It is understood that global consensus is possible

MA Cuiqin
School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China.
Email: cuiqinma@amss.ac.cn.
ZHAO Weiwei
No. 1 Senior Middle School of Qufu, Qufu 273165, China. Email: 904236574@qq.com.
ZHAO Yunbo
College of Information Engineering, Zhejiang University of Technology, Hangzhou 310023, China.
Email: ybzhao@zjut.edu.cn.
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usually by and only by local cooperative interactions among the agents. However, besides cooperation, antagonism can also be popular in real world scenarios. For example, in social groups, antagonism and dislike between pairs of individuals are ubiquitous, which then often leads to a polarization: The whole group is divided into two consensus sub-groups with exactly the opposite opinion[7]. Such a phenomenon is common in many contexts with antagonism, such as the economic systems[8], teams opposed in a sport match, and chaotic systems[9], etc..

Referred to as "bipartite consensus", the above scenario has been carefully addressed by a newly proposed mathematical model in [10]. In the model, the communication networks were modeled as signed graphs in which the edge may have either positive weight (for cooperation) or negative weight (for antagonism) for the interaction between the two agents linked by the edge. Then, the authors proved that first-order integrator MASs with strongly connected signed graph can reach bipartite consensus with some appropriate protocol, if and only if the signed graph was structurally balanced. The notion "bipartite consensus", officially defined as the consensus where the final agent states can be different only in the sign (hence two subgroups as in the earlier example), has become more and more popular since then.

Considerable works have already been found for the development of bipartite consensus for MASs. To name a few, in [11], the signed digraph was relaxed to be having a spanning tree and sufficient conditions were given for bipartite consensus. The extensions to the cases of high-order systems were discussed in [12, 13], whose dynamics were general linear time-invariant systems and single input high-order systems, respectively. Bipartite consensus over time-varying signed graphs was considered in [14]. Despite these existing works, the research on bipartite consensus is still in its infancy. For example, most works focus on the nominal system without the consideration of measurement noise, despite tremendous such works for conventional consensus[15–22]. For some pioneering works for bipartite consensus with measurement noise, the reader may refer to [23, 24] for first-order integrator MASs, and [25] for bipartite linear χ-consensus of double-integrator MASs.

Following this line of research, in the present work we consider the bipartite consensus of discrete-time second-order MASs with a particular focus on the effects of measurement noise, motivated by such applications as the multivehicle system[26]. Sufficient and necessary conditions for mean square bipartite consensus are provided with the help of powerful tools from linear transformation and algebraic graph theory. It turns out that structural balance and having a spanning tree are necessary and sufficient conditions on communication topology to ensure a mean square bipartite consensus. Furthermore, if the signed digraph is also weight balanced, the protocol is a mean square bipartite average consensus protocol, i.e., the mathematical expectation of the position is an average of initial positions and velocities of all agents and both the variance and mathematical expectation of the velocity are zeroes.

This work is challenging due to the following reasons. 1) Similar to [15, 16, 20–22], a decreasing step size $h(k)$ is introduced into the bipartite consensus protocols. It makes the closed-loop system a time-varying stochastic system and hence, analysis tools in [10–14], which are for bipartite consensus without measurement noise, do not work. 2) The continuous-time results in [20] and [25] are obtained by analyzing the state transition matrix of the closed-loop system whose elements are expressed by integrals of exponential functions. Here, in the discrete-time case, the state transition matrix is an infinite product of the state matrices, and thus it is almost impossible to obtain its specific form. 3) In conventional consensus settings, 0 must be an eigenvalue of $\mathcal{L}$ and $\mathcal{L}(1,1,\cdots,1)^{\mathrm{T}}=0$. This property is crucial in analyzing conventional consensus behaviors, but is generally not held for bipartite consensus, meaning the failure of conventional powerful analysis tools[20–22].

The rest of the paper is organized as follows. In Section 2, we formulate the problem based on the preliminaries of signed graph. Then, the main theorems are established in Section 3. Two simulation examples are given in Section 4 to illustrate the theorems. Finally, in Section 5, concluding remarks are provided.

Notations I_n and $\mathbf{0}$ denote the identity and null matrices with required dimensions, respectively. $\mathbf{1}=(1,1,\ \cdots,1)^{\mathrm{T}}$, $\otimes$ is a kronecker product, and $\mathrm{sgn}(\cdot)$ denotes a sign function. $\mathrm{Re}(\lambda)$ denotes the real part of λ. For given random variables x and y, $E(x)$ denotes the mathematical expectation of x, $D(x)$ is its variance and $\mathrm{Cov}(x,y)$ is the covariance of y and x.

2 Preliminaries and Problem Formulation

2.1 Signed Digraph

The communication interactions among agents are described by a signed digraph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ that consists of a vertex set $\mathcal{V}=\{1,2,\cdots,N\}$, an edge set $\mathcal{E}$, and a weighted adjacency matrix $\mathcal{A}=(a_{ij})\in\mathbb{R}^{N\times N}$. $a_{ij}\neq 0\Longleftrightarrow (j,i)\in\mathcal{E}$. a_{ij} can be positive/negative. $a_{ij}<0$ means that agents i and j are antagonistic and $a_{ij}>0$ means that i and j are cooperative. $\mathcal{N}_i=\{j\in\mathcal{V}\mid (j,i)\in\mathcal{E}\}$ is the neighbor set of agent i. Throughout this paper, we always assume that $a_{ii}=0$ and $a_{ij}a_{ji}\geq 0$, $i,j=1,2,\cdots,N$. Let $\mathcal{L}=\mathcal{D}_{\mathcal{G}}-\mathcal{A}$ be Laplacian of $\mathcal{G}$, where

$$\mathcal{D}_{\mathcal{G}}=\mathrm{diag}\left(\sum_{j=1}^{N}|a_{1j}|,\sum_{j=1}^{N}|a_{2j}|,\cdots,\sum_{j=1}^{N}|a_{Nj}|\right).$$

A signed digraph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ is called weight balanced if $\sum_{k=1}^{N}|a_{mk}|=\sum_{k=1}^{N}|a_{km}|$, $m=1,2,\cdots,N$.

For a signed digraph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$, if there exists a bipartition $\{\mathcal{V}_1,\mathcal{V}_2\}$ of $\mathcal{V}$ such that $\mathcal{V}_1\bigcup\mathcal{V}_2=\mathcal{V}$, $\mathcal{V}_1\bigcap\mathcal{V}_2=\varnothing$ and $a_{kl}\geq 0$, for any $k,l\in\mathcal{V}_d$, $d=1,2$; $a_{kl}\leq 0$, for any $k\in\mathcal{V}_d,l\in\mathcal{V}_m$, $d\neq m,d,m=1,2$, then $\mathcal{G}$ is structurally balanced. Otherwise, $\mathcal{G}$ is structurally unbalanced.

Lemma 2.1 (see [27]) *Laplacian $\mathcal{L}$ of $\mathcal{G}$ has at least one zero eigenvalue and all the other eigenvalues have positive real parts if $\mathcal{G}$ is structurally balanced. Moreover, $\mathcal{L}$ has exactly one zero eigenvalue if and only if $\mathcal{G}$ has a spanning tree.*

Lemma 2.2 *If $\mathcal{G}$ is structurally balanced, then $\mathcal{G}$ is weight balanced if and only if $\exists S=\mathrm{diag}(s_1,s_2,\cdots,s_N)$ $(s_i=\pm 1,\ i=1,2,\cdots,N)$ such that SAS has all nonnegative elements and $\mathbf{1}^{\mathrm{T}}S\mathcal{L}=\mathbf{0}$.*

Proof The proof is omitted for simplicity. ∎

2.2 Problem Formulation

The MAS under our consideration is composed of N dynamic agents. The ith agent is described by

$$x_i(k+1) = x_i(k) + v_i(k), \quad v_i(k+1) = v_i(k) + u_i(k), \quad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(k) \in \mathbb{R}$, $v_i(k) \in \mathbb{R}$, $u_i(k) \in \mathbb{R}$ denote the position, velocity and control input of the ith agent, respectively.

To take into account the measurement noise, let $y_{x_{ij}}(k) = x_j(k) + \sigma_{x_{ij}}\eta_{x_{ij}}(k)$, and $y_{v_{ij}}(k) = v_j(k) + \sigma_{v_{ij}}\eta_{v_{ij}}(k)$, $j \in \mathcal{N}_i$ be the position and velocity of the jth agent measured by the ith agent, respectively, where $\{\eta_{x_{ij}}(k), \eta_{v_{ij}}(k),\ i, j = 1, 2, \cdots, N\}$ is the independent standard white noise and $\{\sigma_{x_{ij}} > 0, \sigma_{v_{ij}} > 0,\ i, j = 1, 2, \cdots, N\}$ is the noise intensity.

For the ith agent, we consider the following protocol

$$u_i(k) = -v_i(k) + h(k)\sum_{j\in\mathcal{N}_i}|a_{ij}|\left[\left(\mathrm{sgn}(a_{ij})y_{x_{ij}}(k) - x_i(k)\right) + \left(\mathrm{sgn}(a_{ij})y_{v_{ij}}(k) - v_i(k)\right)\right], \tag{2}$$

where $h(k) > 0$ is the time-varying consensus gain, and $\lim_{k\to\infty} h(k) = 0$.

Denote $X_i(k) = (x_i(k), v_i(k))^{\mathrm{T}}$, and $X(k) = \left(X_1^{\mathrm{T}}(k), X_2^{\mathrm{T}}(k), \cdots, X_N^{\mathrm{T}}(k)\right)^{\mathrm{T}} \in \mathbb{R}^{2N}$. Then substituting (2) into (1) yields the following closed-loop system in the form of a stochastic system:

$$X(k+1) = \left[I_N \otimes \mathfrak{F} + h(k)\mathcal{L} \otimes B\right]X(k) + h(k)\Theta\eta(k), \tag{3}$$

where $\mathfrak{F} = \left(\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix}0 & 0\\ -1 & -1\end{smallmatrix}\right)$, $\Theta = \mathrm{diag}(\Theta_1, \Theta_2, \cdots, \Theta_N) \in \mathbb{R}^{2N\times 2N^2}$, $\Theta_i = (a_{i1}\left(\begin{smallmatrix}0 & 0\\ \sigma_{x_{i1}} & \sigma_{v_{i1}}\end{smallmatrix}\right),$ $a_{i2}\left(\begin{smallmatrix}0 & 0\\ \sigma_{x_{i1}} & \sigma_{v_{i2}}\end{smallmatrix}\right), \cdots, a_{iN}\left(\begin{smallmatrix}0 & 0\\ \sigma_{x_{iN}} & \sigma_{v_{iN}}\end{smallmatrix}\right)) \in \mathbb{R}^{2\times 2N}, i = 1, 2, \cdots, N$, $\eta(k) = (\eta_{x_{11}}(k), \eta_{v_{11}}(k), \cdots,$ $\eta_{x_{1N}}(k), \eta_{v_{1N}}(k), \cdots, \eta_{x_{N1}}(k), \eta_{v_{N1}}(k), \cdots, \eta_{x_{NN}}(k), \eta_{v_{NN}}(k))^{\mathrm{T}} \in \mathbb{R}^{2N^2}$.

Definition 2.3 A distributed protocol $\mathcal{U} = \{u_i,\ i = 1, 2, \cdots, N\}$ is said to be a mean square bipartite consensus protocol for the system in (1), if for any given $X_i(k_0) \in \mathbb{R}^2,\ k_0 \geq 0$, there exist ξ_x^* and ξ_v^* such that

$$\lim_{k\to\infty} E[x_i(k) - \gamma_i\xi_x^*]^2 = 0, \quad \lim_{k\to\infty} E[v_i(k) - \gamma_i\xi_v^*]^2 = 0, \tag{4}$$

where $\gamma_i \in \{\pm 1\}$, $(i = 1, 2, \cdots, N)$, $E(\xi_v^*) = D(\xi_v^*) = 0$, $E(\xi_x^*) = \psi(x_1(k_0), v_1(k_0), \cdots, x_N(k_0),$ $v_N(k_0))$, $D(\xi_x^*) < \infty$, and $\psi(\cdot)$ is a linear function.

The above definition can be compared with conventional consensus[16] and continuous-time bipartite consensus[25], respectively. Intuitively, the conditions in (4) mean that each agent's position and velocity will converge in mean square sense to $\pm\xi_x^*$ and $\pm\xi_v^*$, respectively.

As mentioned earlier the interactions among the agents in (1) are illustrated by a signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. In what follows, we list the following conditions for better reading.

($\mathbf{H_1}$) Signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ has a spanning tree.

($\mathbf{H_2}$) Signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced.

($\mathbf{H_3}$) $\sum_{i=0}^{\infty} h(i) = \infty$.

($\mathbf{H_4}$) $\sum_{i=0}^{\infty} h^2(i) < \infty$.

($\mathbf{H_5}$) $\mathcal{G}$ is weight balanced.

Remark 2.4 In Definition 2.3, if $\psi(\cdot)$ is a weight average of the initial states, i.e., $E(\xi_x^*) = \frac{1}{N}\sum_{i=1}^{N}\gamma_i[x_i(k_0) + v_i(k_0)]$, then the protocol $\mathcal{U}$ is called a mean square bipartite average consensus protocol.

Remark 2.5 The protocol in (2) is consistent with the protocol in [20] if $a_{ij} \geq 0$, $i, j = 1, 2, \cdots, N$, i.e., the signed digraph reduces to a conventional graph with nonnegative weights. In (2), $h(k)$ is needed to be sufficiently small when k is sufficiently large, that is, $\lim_{k\to\infty} h(k) = 0$. This is to make sure that the eigenvalues of the state matrix in the closed-loop system are in the unit circle of the complex plane.

Remark 2.6 From the subsequent analysis, it can be seen that $\gamma_i \in \{\pm 1\}$ in Definition 2.3 is determined by the communication topology and independent of the initial states.

2.3 Useful Lemmas

Lemma 2.7 *Consider the following linear time-varying discrete-time system*

$$y(k+1) = J_q(k,\lambda)y(k), \tag{5}$$

where $q \in \mathbb{N}, \lambda \in \mathbb{C}, Re(\lambda) > 0, y(k) = (y_{11}(k), y_{12}(k), \cdots, y_{q1}(k), y_{q2}(k))^{\mathrm{T}} \in \mathbb{R}^{2q}$,

$$J_q(k,\lambda) = \begin{pmatrix} J_q^1 & J_q^2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & J_q^1 & J_q^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & J_q^1 \end{pmatrix}_{2q\times 2q},$$

$$J_q^1 = \begin{pmatrix} 1 & 1 \\ -\lambda h(k) & -\lambda h(k) \end{pmatrix}, \quad J_q^2 = \begin{pmatrix} 0 & 0 \\ -h(k) & -h(k) \end{pmatrix}.$$

Assume $\Phi^\lambda(k,k_0)$ *is the state transition matrix of* (5). *Then* $\lim_{k\to\infty} \Phi^\lambda(k,k_0) = \mathbf{0}$, *if* $\lim_{k\to\infty} h(k) = 0$ *and* $\sum_{i=0}^{\infty} h(i) = \infty$.

Proof Denote $\overline{y}_i(k) = y_{i1}(k) + y_{i2}(k)$, $i = 1, 2, \cdots, q$, and define $\overline{y}(k) = (\overline{y}_1(k), \overline{y}_2(k), \cdots, \overline{y}_q(k))^{\mathrm{T}} \in \mathbb{R}^q$. Then it can be easily shown that

$$\overline{y}(k+1) = \overline{J}_q(k,\lambda)\overline{y}(k), \tag{6}$$

where

$$\overline{J}_q(k,\lambda) = \begin{pmatrix} 1-\lambda h(k) & -h(k) & \cdots & 0 \\ 0 & 1-\lambda h(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\lambda h(k) \end{pmatrix}_{q\times q}.$$

Let $\overline{\Phi}^{\lambda}(k,k_0)$ be the state transition matrix of the system in (6). Then

$$\overline{y}(k)=\overline{\Phi}^{\lambda}(k,k_0)\overline{y}(k_0). \tag{7}$$

According to Lemma 5 of [28], for any $\varepsilon>0$, there exists $M>0$ such that $\|\overline{\Phi}^{\lambda}(k,k_0)\|_2\leq M\mathrm{e}^{-\frac{\mathrm{Re}(\lambda)-\varepsilon}{2}\sum_{i=k_0}^{k}h(i)}$, if $\lim_{k\to\infty}h(k)=0$. It follows that $\lim_{k\to\infty}\overline{\Phi}^{\lambda}(k,k_0)=\mathbf{0}$, if $\sum_{i=0}^{\infty}h(i)=\infty$. Combining this with (7) gives $\lim_{k\to\infty}\overline{y}(k)=\mathbf{0}$, i.e., $\lim_{k\to\infty}\overline{y}_i(k)=0, i=1,2,\cdots,q$. Recall $\overline{y}_i(k)=y_{i1}(k)+y_{i2}(k)$, $i=1,2,\cdots,q$. By (5), one obtains that $y_{i1}(k+1)=\overline{y}_i(k), i=1,2,\cdots,q$, $y_{i2}(k+1)=-\lambda h(k)\overline{y}_i(k)-h(k)\overline{y}_{i+1}(k), i=1,2,\cdots,q-1$, and $y_{q2}(k+1)=-\lambda h(k)\overline{y}_q(k)$. This together with $\lim_{k\to\infty}h(k)=0$ leads to $\lim_{k\to\infty}y(k)=\mathbf{0}$. Since $y(k)=\Phi^{\lambda}(k,k_0)y(k_0)$, by the arbitrariness of $y(k_0)$, $\lim_{k\to\infty}\Phi^{\lambda}(k,k_0)=\mathbf{0}$. ∎

The state transition matrix reveals many essential properties of the closed-loop system in (3). The next lemma for the state transition matrix is indispensable to the convergence analysis of the closed-loop system in (3).

Lemma 2.8 *If* (2) *is a mean square bipartite consensus protocol, then there exists $v\in\mathbb{R}^{2N}$, such that*

$$\lim_{k\to\infty}\Phi(k,k_0)=\beta v^{\mathrm{T}},$$

where $\beta=(\gamma_1,0,\gamma_2,0,\cdots,\gamma_N,0)^{\mathrm{T}}\in\mathbb{R}^{2N}$.

Proof From Definition 2.3 we know that for any given $X_i(k_0)\in\mathbb{R}^2$, $i=1,2,\cdots,N,k_0\geq 0$, there exist random variables ξ_x^* and ξ_v^* such that

$$\lim_{k\to\infty}E[x_i(k)-\gamma_i\xi_x^*]^2=0,\qquad \lim_{k\to\infty}E[v_i(k)-\gamma_i\xi_v^*]^2=0,$$

where $\gamma_i=\pm1$, $i=1,2,\cdots,N$, $E(\xi_x^*)=\frac{1}{N}\sum_{i=1}^{N}\gamma_i[x_i(k_0)+v_i(k_0)]$, and $E(\xi_v^*)=0$.

Denote $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_N)^{\mathrm{T}}$. Then

$$\lim_{k\to\infty}E\|X(k)-\gamma\otimes(\xi_x^*,\xi_v^*)^{\mathrm{T}}\|^2=0. \tag{8}$$

Let $\Phi(k,k_0)$ be the state transition matrix of (3). Then the solution of (3) can be easily shown as $X(k)=\Phi(k,k_0)X(k_0)+\sum_{i=k_0}^{k-1}\Phi(k,i+1)h(i)\Theta\eta(i)$. Define $X_2(k)\triangleq\sum_{i=k_0}^{k-1}\Phi(k,i+1)h(i)\Theta\eta(i)$, and assume that $X_2(k)$ converges to X_2^* in mean square. Then, by (8), one has

$$\gamma\otimes(E\xi_x^*,0)^{\mathrm{T}}=\lim_{k\to\infty}EX(k)=\lim_{k\to\infty}\Phi(k,k_0)X(k_0)+EX_2^*. \tag{9}$$

Since $X(k_0)$ is arbitrary, there exist $\overline{\gamma}$, $\overline{\xi^*}$ and $\widehat{\gamma}$, $\widehat{\xi^*}$, respectively, such that

$$\begin{aligned}\overline{\gamma}\otimes(E\overline{\xi^*},0)^{\mathrm{T}}&=2\lim_{k\to\infty}\Phi(k,k_0)X(k_0)+EX_2^*,\\ \widehat{\gamma}\otimes(E\widehat{\xi^*},0)^{\mathrm{T}}&=3\lim_{k\to\infty}\Phi(k,k_0)X(k_0)+EX_2^*,\end{aligned} \tag{10}$$

where $\overline{\gamma}_i,\widehat{\gamma}_i\in\{\pm1\}$, $i=1,2,\cdots,N$. This together with (9) leads to

$$\gamma\otimes(E\xi_x^*,0)^{\mathrm{T}}=\overline{\gamma}\otimes 2(E\overline{\xi^*},0)^{\mathrm{T}}-\widehat{\gamma}\otimes(E\widehat{\xi^*},0)^{\mathrm{T}}. \tag{11}$$

Immediately it follows that $\gamma = \pm\overline{\gamma} = \pm\widehat{\gamma}$.

By (10), we assume that $\lim_{k\to\infty} \Phi(k,k_0)X(k_0) = \gamma \otimes (f_X^*, 0)^{\mathrm{T}}$. Since $X(k_0)$ is arbitrary, one can take $X(k_0)$ as $e_i = (0,\cdots,0,\underbrace{1}_{i},0,\cdots,0)^{\mathrm{T}}, i = 1,\cdots,2N$, respectively. Immediately it follows that

$$\begin{aligned}\lim_{k\to\infty} \Phi(k,k_0) &= \left(\gamma\otimes(f_{e_1}^*,0)^{\mathrm{T}}, \gamma\otimes(f_{e_2}^*,0)^{\mathrm{T}},\cdots,\gamma\otimes(f_{e_{2N}}^*,0)^{\mathrm{T}}\right)\\ &= (\gamma_1,0,\gamma_2,0,\cdots,\gamma_N,0)^{\mathrm{T}}(f_{e_1}^*,f_{e_2}^*,\cdots,f_{e_{2N}}^*)_{1\times 2N} \triangleq \beta v^{\mathrm{T}}.\end{aligned}$$

This completes the proof. ∎

Remark 2.9 From the proof of Lemma 2.8 we know that γ in Definition 2.3 may differ at most by a minus sign for any given initial values. In this sense, γ can be considered the same since the minus can be seen as a part of random variable ξ_x^*/ξ_v^*. Therefore, γ is independent of initial values.

3 Mean Square Bipartite Consensus

Theorem 3.1 *Given the system in* (1), (2) *is a mean square bipartite consensus protocol if and only if Conditions* (H_1)–(H_4) *hold.*

Proof Sufficiency. (H_1)–(H_2) and Lemma 2.1 imply that 0 is a simple eigenvalue of $\mathcal{L}$ and all the other eigenvalues have positive real parts. Then there must exist an invertible matrix $C \in \mathbb{C}^{N\times N}$, such that

$$C^{-1}\mathcal{L}C = \Lambda = \mathrm{diag}(0, J_2, J_3, \cdots, J_s), \tag{12}$$

where J_i is q_i dimensional Jordan block with eigenvalue λ_i on its diagonal. Obviously, $q_2+q_3+\cdots+q_s = N-1$ and $\mathrm{Re}(\lambda_i) > 0,\ i = 2,3,\cdots,s$. Therefore, the closed-loop system in (3) can be rewritten as

$$X(k+1) = (C\otimes I_2)\mathrm{diag}\left(\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, J_{q_2}(k,\lambda_2),\cdots,J_{q_s}(k,\lambda_s)\right)(C^{-1}\otimes I_2)X(k) + h(k)\Theta\eta(k),$$

where $J_{r_m}(k,\lambda_m),\ m = 2,3,\cdots,s$ is defined as in (6). As follows from Lemma 2.7, the state transition matrix $\Phi(k,k_0)$ of (3) satisfies

$$\Phi(k,k_0) = (C\otimes I_2)\mathrm{diag}\left(\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \Phi^{\lambda_2}(k,k_0),\cdots,\Phi^{\lambda_s}(k,k_0)\right)(C^{-1}\otimes I_2),$$

where $\Phi^{\lambda_m}(k,k_0),\ m = 2,3,\cdots,s$ is defined as in Lemma 2.7. Immediately from Lemma 2.7, (H_3) and (H_4), one obtains that $\lim_{k\to\infty}\Phi^{\lambda_m}(k,k_0) = \mathbf{0}, m = 2,3,\cdots,s$. Therefore,

$$\lim_{k\to\infty}\Phi(k,k_0) = (C\otimes I_2)\mathrm{diag}\left(\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \mathbf{0},\cdots,\mathbf{0}\right)(C^{-1}\otimes I_2). \tag{13}$$

The next step is to prove $X_2(k) \triangleq \sum_{i=k_0}^{k-1} \Phi(k,i+1)h(i)\Theta\eta(i)$ converges in mean square to a random variable by using Cauchy convergence criterion[29]. From (H_4) we know that for any $\varepsilon > 0$, $\exists K_1 > 0$ satisfying

$$\sum_{i=K_1}^{\infty} h^2(i) < \varepsilon. \tag{14}$$

By (13), for the given $\varepsilon > 0$, $\exists K_2 > K_1$ such that for $\forall l_1 \geq l_2 > K_2$,

$$\|\Phi(l_1,i) - \Phi(l_2,i)\|_2 < \varepsilon, \qquad \forall i = k_0, k_0+1, \cdots, K_1. \tag{15}$$

By (13), there exists $\mathfrak{M}_\Phi < \infty$ satisfying

$$\|\Phi(\cdot,\cdot)\|_2 \leq \mathfrak{M}_\Phi. \tag{16}$$

Since

$$X_2(l_1) - X_2(l_2) = \sum_{j=k_0}^{l_2-1}\Big[\Phi(l_1,j+1) - \Phi(l_2,j+1)\Big]h(j)\Theta\eta(j) + \sum_{j=l_2}^{l_1-1}\Phi(l_1,j+1)h(j)\Theta\eta(j),$$

by (14), (15) and (16), one has

$$\left\|E\left[\Big(X_2(l_1) - X_2(l_2)\Big)\Big(X_2(l_1) - X_2(l_2)\Big)^{\mathrm{T}}\right]\right\|_2 \leq \varepsilon^2 a \sum_{j=k_0}^{K_1-1} h^2(i) + 4\mathfrak{M}_\Phi^2 \varepsilon a + \mathfrak{M}_\Phi^2 \varepsilon a,$$

where $a = \|\Theta\Theta^{\mathrm{T}}\|_2$. Since ε is arbitrary, there must exist a random vector $\mathfrak{X}_2^*$ such that $X_2(k)$ converges in mean square to $\mathfrak{X}_2^*$ by Cauchy convergence criterion. Thus, $X(k) = \Phi(k,k_0)X(k_0) + X_2(k)$ converges to $\mathfrak{X}^*$ in mean square sense. Denote $\mathfrak{X}^* \triangleq (\mathfrak{x}_1^*, \mathfrak{v}_1^*, \mathfrak{x}_2^*, \mathfrak{v}_2^*, \cdots, \mathfrak{x}_N^*, \mathfrak{v}_N^*)^{\mathrm{T}}$. Then, by (13), one obtains

$$\mathfrak{X}^* = (C \otimes I_2)\mathrm{diag}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{0}, \cdots, \mathbf{0}\right)(C^{-1} \otimes I_2)X(k_0) + \mathfrak{X}_2^*.$$

Thus,

$$E(\mathfrak{X}^*) = (C \otimes I_2)\mathrm{diag}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{0}, \cdots, \mathbf{0}\right)(C^{-1} \otimes I_2)X(k_0). \tag{17}$$

Denote the first column of C as C_r and define the first row of C^{-1} as $C_b^{\mathrm{T}} = (b_1, b_2, \cdots, b_N)$. Then $C_b^{\mathrm{T}} C_r = 1$. By (12), $\mathcal{L}C = C\Lambda$ and $C^{-1}\mathcal{L} = \Lambda C^{-1}$. Thus, $C_b^{\mathrm{T}}\mathcal{L} = \mathbf{0}$ and $\mathcal{L}C_r = \mathbf{0}$. From ($H_2$) and Lemma 1 of [10] we know that there exists $F = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_N)$ ($\mu_i \in \{\pm 1\}$) such that $F\mathcal{A}F$ has all nonnegative elements, and hence, $F\mathcal{L}F\mathbf{1} = \mathbf{0}$. Since 0 is a simple eigenvalue of $\mathcal{L}$, its eigensubspace is one dimensional. Without loss of generality we may assume that $C_r =$

$F\mathbf{1} = (\mu_1, \mu_2, \cdots, \mu_N)^{\mathrm{T}}$. This together with (17) gives $E(\mathfrak{x}_i^*) = \mu_i \sum_{j=1}^N b_j[x_j(k_0) + v_j(k_0)]$, $E(\mathfrak{v}_i^*) = 0$, $i = 1, 2, \cdots, N$. By definition,

$$\begin{aligned} D(\mathfrak{X}^*) &\triangleq E\left[(\mathfrak{X}^* - E(\mathfrak{X}^*))(\mathfrak{X}^* - E(\mathfrak{X}^*))^{\mathrm{T}}\right] \\ &= E\left[\mathfrak{X}_2^*(\mathfrak{X}_2^*)^{\mathrm{T}}\right] \\ &= \lim_{k\to\infty} \sum_{l=k_0}^{k-1} h^2(l)\Phi(k, l+1)\Theta E\left[\eta(l)\eta^{\mathrm{T}}(l)\right] \Theta^{\mathrm{T}} \Phi^{\mathrm{T}}(k, l+1). \end{aligned} \tag{18}$$

Denote

$$\Phi^* = (C \otimes I_2)\mathrm{diag}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, 0, \cdots, 0 \right)(C^{-1} \otimes I_2).$$

Then following the similar procedure as in Theorem 1 of [20], one has

$$D(\mathfrak{X}^*) = \sum_{l=k_0}^{\infty} h^2(l)\Phi^*\Theta\Theta^{\mathrm{T}}(\Phi^*)^{\mathrm{T}} = (\Delta_{kl})_{N\times N} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\Delta_{kl} = \mu_k\mu_l\Delta$, $k, l = 1, 2, \cdots, N$, $\Delta = \sum_{l=k_0}^{\infty} h^2(l) \sum_{l=1}^N \sum_{j=1}^N b_l^2 a_{lj}^2(\sigma_{x_{lj}}^2 + \sigma_{v_{lj}}^2)$. Immediately, one has $D(\mathfrak{x}_l^*) = \Delta_{ll} = \Delta < \infty$, $D(\mathfrak{v}_l^*) = 0$, $\mathrm{Cov}(\mathfrak{x}_l^*, \mathfrak{x}_m^*) = \Delta_{lm}$ and $\mathrm{Cov}(\mathfrak{v}_l^*, \mathfrak{v}_m^*) = 0, \forall l, m = 1, 2, \cdots, N$. Hence, $\forall m = 1, 2, \cdots, N$,

$$\begin{aligned} &\lim_{k\to\infty} E\left[x_m(k) - \mu_m\mu_1\mathfrak{x}_1^*\right]^2 \\ &\leq \lim_{k\to\infty} E\left[x_m(k) - \mathfrak{x}_m^*\right]^2 + E\left[\mathfrak{x}_m^* - \mu_m\mu_1\mathfrak{x}_1^*\right]^2 + 2\lim_{k\to\infty}\left[E(x_m(k) - \mathfrak{x}_m^*)^2\right]^{\frac{1}{2}}\left[E(\mathfrak{x}_m^* - \mu_m\mu_1\mathfrak{x}_1^*)^2\right]^{\frac{1}{2}} \\ &= D\left(\mathfrak{x}_m^*\right) + \left(E\mathfrak{x}_m^*\right)^2 + D\left(\mathfrak{x}_1^*\right) + (E\mathfrak{x}_1^*)^2 - 2\mu_m\mu_1[\Delta_{m1} + (E\mathfrak{x}_m^*)(E\mathfrak{x}_1^*)] \\ &= 0. \end{aligned}$$

Similarly, $\lim_{k\to\infty} E[v_m(k) - \mu_m\mu_1\mathfrak{v}_1^*]^2 = 0$. Taking $\xi_x^* = \mathfrak{x}_1^*$, $\xi_v^* = \mathfrak{v}_1^*$ and $\gamma_m = \mu_m\mu_1$, $m = 1, 2, \cdots, N$, by Definition 2.3, one gets the sufficiency.

Necessity. The necessity is proved in the following five steps.

Step (I) Prove the necessity of (H_3), i.e., $\sum_{i=0}^{\infty} h(i) = \infty$.

Assume by contradiction that (H_3) does not hold, i.e., $\sum_{i=0}^{\infty} h(i) < \infty$. Then, there exists $M > 0$ such that $\sum_{i=0}^{\infty} h(i) = M$. Denote $\overline{X}(k) = [I_N \otimes (1, 1)]\, X(k) \triangleq (\overline{x}_1(k), \overline{x}_2(k), \cdots, \overline{x}_N(k))^{\mathrm{T}} \in \mathbb{R}^N$. Then, by (3),

$$\overline{X}(k+1) = [I_N - h(k)\mathcal{L}]\, \overline{X}(k) + h(k)\, [I_N \otimes (1, 1)]\, \Theta\eta(k). \tag{19}$$

Let $\overline{\Phi}(k, k_0)$ be the state transition matrix of the system in (19). Then, $\overline{\Phi}(k, k_0) = \prod_{i=k_0}^{k-1} (I_N - h(i)\mathcal{L})$. By the definition of Laplacian, all the eigenvalues of $\mathcal{L}$ have nonnegative real parts. Thus there exists an invertible matrix P such that $P^{-1}\mathcal{L}P = \mathrm{diag}\,(J_1, J_2, \cdots, J_s)$, where $J_j (1 \leqslant j \leqslant s)$ is an $n_j \times n_j$ Jordan block with λ_j on its diagonal and $n_1 + n_2 + \cdots + n_s = N$. Clearly, $\mathrm{Re}(\lambda_j) \geq 0, j = 1, 2, \cdots, s$. Therefore, $\overline{\Phi}(k, k_0) = P\prod_{i=k_0}^{k-1} (I_N - h(i)J)\, P^{-1}$. Since $\sum_{i=0}^{\infty} h(i) < \infty$, $\lim_{k\to\infty} h(k) = 0$. Therefore, there exists $N^* > 0$ such that for any $k > N^*$,

$0 \le \lambda^* h(k) < \frac{1}{2}\ln 2$, where $\lambda^* = \max\{\mathrm{Re}(\lambda_j), j = 1, 2, \cdots, s\}$. Taking $k_0 = N^* + 1$, by $1 - x \ge \mathrm{e}^{-2x}$ $(0 \le x < \frac{1}{2}\ln 2)$, one obtains that for any $j = 1, 2, \cdots, s$,

$$\begin{aligned}\left|\prod_{i=k_0}^{k-1}(1-\lambda_j h(i))\right| &= \prod_{i=k_0}^{k-1}|1-\lambda_j h(i)| \ge \prod_{i=k_0}^{k-1}|1-\mathrm{Re}(\lambda_j)h(i)| \\ &\ge \prod_{i=k_0}^{k-1}\mathrm{e}^{-2\lambda^* h(i)} = \mathrm{e}^{-2\lambda^*\sum_{i=k_0}^{k-1}h(i)} \ge \mathrm{e}^{-2\lambda^* M} > 0.\end{aligned}$$

Therefore, $\lim_{k\to\infty}\left[\prod_{i=k_0}^{k-1}(1-\lambda_j h(i))\right] \ne 0$, and hence $\mathrm{rank}\left[\lim_{k\to\infty}\overline{\Phi}(k,k_0)\right] = N$. Since $\Phi(k,k_0)$ is the state transition matrix of (3), $\overline{\Phi}(k,k_0)[I_N \otimes (1,1)] = [I_N \otimes (1,1)]\Phi(k,k_0)$. Immediately, one has $I_N \otimes (1,1) = \left(\lim_{k\to\infty}\overline{\Phi}(k,k_0)\right)^{-1}[I_N \otimes (1,1)]\left(\lim_{k\to\infty}\Phi(k,k_0)\right)$. Hence, $\mathrm{rank}\left[\lim_{k\to\infty}\Phi(k,k_0)\right] = N$. This contradicts the statement of Lemma 2.8. Thus, $\sum_{i=0}^{\infty}h(i) = \infty$, i.e., (H_3) holds.

Step (II) Prove that 0 is a simple eigenvalue of $\mathcal{L}$.

First, we prove that 0 is an eigenvalue of $\mathcal{L}$. If otherwise, 0 is not an eigenvlaue of $\mathcal{L}$. Then from (H_3) and Lemma 2.7 we know that $\lim_{k\to\infty}\Phi(k,k_0) = \mathbf{0}$. Combining this with (9) leads to $EX_2^* = \gamma\otimes(E\xi_x^*, 0)^{\mathrm{T}}$. Since EX_2^* has nothing to do with the initial state $X(k_0)$, $E\xi_x^*$ is independent of $X(k_0)$. This contradicts the statement that $E\xi_x^* = \psi(x_1(k_0), v_1(k_0), \cdots, x_N(k_0), v_N(k_0))$. Thus, 0 is an eigenvalue of $\mathcal{L}$. Assume J_1^0 is a Jordan block associated with eigenvalue 0. Then J_1^0 must be one dimensional. If otherwise, without loss of generality we may assume that J_1^0 is two dimensional, i.e., $J_1^0 = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$. A direct calculation gives

$$\Phi^0(k,k_0) = \begin{pmatrix}1 & 1 & -\sum_{i=k_0+1}^{k-1}h(i) & -\sum_{i=k_0+1}^{k-1}h(i)\\ 0 & 0 & -h(k_0) & -h(k_0)\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix},$$

which is a block matrix associated with J_1^0 in $\Phi(k,k_0)$. By $\sum_{i=0}^{\infty}h(i) = \infty$, $\lim_{k\to\infty}\Phi^0(k,k_0)$ does not exist and hence $\lim_{k\to\infty}\Phi(k,k_0)$ does not exist. This contradicts Lemma 2.8. Other cases can be similarly proved. Therefore, the Jordan block corresponding to eigenvalue 0 is one dimensional.

Second, we prove 0 is a simple eigenvalue of $\mathcal{L}$. Assume by contradiction that the multiplicity of 0 is m and $m > 1$. Without loss of generality we may assume that $m = 2$. Since the Jordan block corresponding to 0 is one dimensional, by a abuse of notation,

$$\lim_{k\to\infty}\Phi(k,k_0) = (C\otimes I_2)\mathrm{diag}\left(\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \mathbf{0}, \cdots, \mathbf{0}\right)(C^{-1}\otimes I_2).$$

Thus, $\mathrm{rank}\left(\lim_{k\to\infty}\Phi(k,k_0)\right) = 2$. This contradicts the statement that $\mathrm{rank}\left(\lim_{k\to\infty}\Phi(k,k_0)\right) = \mathrm{rank}(\beta v^{\mathrm{T}}) \le 1$. Other cases can be similarly proved. So $m = 1$.

Finally, 0 is a simple eigenvalue of $\mathcal{L}$.

Step (III) Prove the necessity of (H_2).

From Step (II) we know that 0 is a simple eigenvalue of $\mathcal{L}$. For simplicity, we use the same symbol in (12) and have $C^{-1}\mathcal{L}C = \Lambda = \mathrm{diag}(0, J_2, J_3, \cdots, J_s)$. Combining (13) with Lemma 2.8 leads to

$$(C \otimes I_2)\mathrm{diag}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{0}, \cdots, \mathbf{0}\right) \times (C^{-1} \otimes I_2) = \beta v^{\mathrm{T}}.$$

This implies that $C_r = (\gamma_1, \gamma_2, \cdots, \gamma_N)^{\mathrm{T}} \cdot f^*$, where C_r is the first column of C and $f^* = v^{\mathrm{T}}\left(C_r \otimes (1\ \ 0)^{\mathrm{T}}\right)$. From (12) we know $\mathcal{L}C_r = \mathbf{0}$. Thus, $\mathcal{L}(\gamma_1, \gamma_2, \cdots, \gamma_N)^{\mathrm{T}} = \mathbf{0}$. Therefore, for $\forall j$, $\gamma_i \sum_{j\neq i} |a_{ij}| = \sum_{j\neq i} \gamma_j a_{ij}, j = 1, 2, \cdots, N$. Since $\gamma_i = \pm 1, \gamma_i^2 = 1, i = 1, 2, \cdots, N$, $\sum_{j\neq i} |a_{ij}| = \sum_{j\neq i} \gamma_i\gamma_j a_{ij}$, and hence $\gamma_i\gamma_j a_{ij} = |a_{ij}| \geq 0$. Denote $\mathcal{V}_1 = \{i|\gamma_i = 1, i = 1, 2, \cdots, N\}$ and $\mathcal{V}_2 = \{i|\gamma_i = -1, i = 1, 2, \cdots, N\}$. Then $\mathcal{V}_1 \bigcup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \bigcap \mathcal{V}_2 = \varnothing$ and for any $k \in \mathcal{V}_s, l \in \mathcal{V}_m, s \neq m, s, m \in \{1, 2\}$, $a_{kl} \leq 0$; for any $k, l \in \mathcal{V}_m, m \in \{1, 2\}$,$a_{kl} \geq 0$. By definition, $\mathcal{G}$ is structurally balanced, i.e., (H_2) holds.

Step (IV) Prove the necessity of (H_1).

Since Step (II) and Step (III) hold, Lemma 2.1 implies that $\mathcal{G}$ has a spanning tree, i.e., (H_1) holds.

Step (V) Prove the necessity of (H_4).

By a slight abuse of notation, we assume $C_b^{\mathrm{T}} = (b_1, b_2, \cdots, b_N)$ is the first row of C^{-1} and $C_b^{\mathrm{T}}\mathcal{L} = \mathbf{0}$. By (3), one has $\left[C_b^{\mathrm{T}} \otimes (1, 1)\right] X(k+1) = \left[C_b^{\mathrm{T}} \otimes (1, 1)\right] X(k) + h(k)\left[C_b^{\mathrm{T}} \otimes (1, 1)\right] \Theta\eta(k)$. Denote $\zeta^{\mathrm{T}} = C_b^{\mathrm{T}} \otimes (1, 1)$. Then $\zeta^{\mathrm{T}} X(k) = \zeta^{\mathrm{T}} X(k_0) + \sum_{i=k_0}^{k-1} h(i)\zeta^{\mathrm{T}}\Theta\eta(i)$. By Definition 2.3 we know that $\sum_{i=k_0}^{k-1} h(i)\zeta^{\mathrm{T}}\Theta\eta(i)$ converges in mean square to a random vector with finite variance. Hence $\lim_{k\to\infty} E\left[\sum_{i=k_0}^{k-1} h(i)\zeta^{\mathrm{T}}\Theta\eta(i)\right]^2 < \infty$. If ($H_4$) does not hold, i.e., $\sum_{i=0}^{\infty} h^2(i) = \infty$, then $\lim_{k\to\infty} E\left[\sum_{i=k_0}^{k-1} h(i)\zeta^{\mathrm{T}}\Theta\eta(i)\right]^2 = \lim_{k\to\infty}\sum_{i=k_0}^{k-1} h^2(i)\zeta^{\mathrm{T}}\Theta\Theta^{\mathrm{T}}\zeta = \infty$. This leads to a contradiction. Hence, $\sum_{i=0}^{\infty} h^2(i) < \infty$, i.e., ($H_4$) holds. ∎

Remark 3.2 From the sufficiency proof of Theorem 3.1 we know that $\gamma_i = \mu_i\mu_1$, $i = 1, 2, \cdots, N$, where μ_i is the element of right eigenvector associated with eigenvalue 0 of $\mathcal{L}$. Therefore, γ_i is determined by the communication topology and has nothing to do with the initial state $X(k_0)$.

Remark 3.3 Under (H_1) and (H_2), the bipartite consensus problem over signed digraphs can be transformed into a consensus problem over traditional digraphs. Therefore, part of the sufficiency proof of Theorem 3.1 can be similarly derived from [20]. In the necessity proof of Theorem 3.1, however, (H_1) and (H_2) are no longer prerequisites, but conclusions to be derived. Thus, methods in [16] and [20] are not applicable.

Remark 3.4 It is worth noting that results in [22] depends on the precondition (H_1), i.e., $\mathcal{G}$ has a spanning tree. Here, from Theorem 3.1, it can be seen that (H_1) is also necessary for mean square bipartite consensus.

Theorem 3.5 *The protocol in* (2) *is a mean square bipartite average consensus protocol for* (1) *if and only if* (H_1)–(H_5) *hold.*

Proof Sufficiency. Since (H_1)–(H_4) hold, using the same arguments as in the sufficiency proof of Theorem 3.1, one sees that 0 is a simple eigenvalue of $\mathcal{L}$, $F\mathcal{L}F\mathbf{1} = \mathbf{0}$, and $C_b = (b_1, b_2, \cdots, b_N)^{\mathrm{T}}$, $C_r = F\mathbf{1} = (\mu_1, \mu_2, \cdots, \mu_N)^{\mathrm{T}}$ ($\mu_i = \pm 1, i = 1, 2, \cdots, N$) are the left and right eigenvector associated with eigenvalue 0, respectively. Applying (H_5), we obtain $\mathbf{1}^{\mathrm{T}}F\mathcal{L}F = \mathbf{0}$. Thanks to $C_b^{\mathrm{T}}C_r = 1$, $C_b^{\mathrm{T}} = \frac{1}{N}\mathbf{1}^{\mathrm{T}}F = \frac{1}{N}(\mu_1, \mu_2, \cdots, \mu_N)$, i.e., $b_j = \frac{1}{N}\mu_j, j = 1, 2, \cdots, N$. By Definition 2.3, the protocol in (2) is a mean square bipartite average consensus protocol.

Necessity. From the necessity proof of Theorem 3.1 it can be seen that (H_1)–(H_4) hold. It suffices to show that (H_5) holds. From Step (III) and Step (V) in the necessity proof of Theorem 3.1, we know that $C_b = (b_1, b_2, \cdots, b_N)^{\mathrm{T}}$ and C_r are the left and right eigenvectors associated with eigenvalue 0 of $\mathcal{L}$, respectively. Furthermore, $C_r = f^*(\gamma_1, \gamma_2, \cdots, \gamma_N)^{\mathrm{T}}$, $\gamma_i\gamma_j a_{ij} \geq 0$. Since $C_b^{\mathrm{T}}C_r = 1$, $f^*\sum_{j=1}^N b_j\gamma_j = 1$. Denote $\zeta^{\mathrm{T}} = C_b^{\mathrm{T}} \otimes (1, 1) = (b_1, b_1, b_2, b_2, \cdots, b_N, b_N)^{\mathrm{T}}$. Then, by (3), one has $\zeta^{\mathrm{T}}X(k) = \zeta^{\mathrm{T}}X(k_0) + \sum_{i=k_0}^{k-1} h(i)\zeta^{\mathrm{T}}\Theta\eta(i)$. Thus, $\lim_{k\to\infty} E\left(\zeta^{\mathrm{T}}X(k)\right) = \zeta^{\mathrm{T}}X(k_0) = \sum_{i=1}^N b_i[x_i(k_0) + v_i(k_0)]$. On the other hand, by Definition 2.3, one has

$$\begin{aligned}\lim_{k\to\infty} E(\zeta^{\mathrm{T}}X(k)) &= \lim_{k\to\infty} E[b_1x_1(k) + b_1v_1(k) + \cdots + b_Nx_N(k) + b_Nv_N(k)] \\ &= \left(\sum_{j=1}^N b_j\gamma_j\right) E(\xi_x^*) \\ &= \left(\sum_{j=1}^N b_j\gamma_j\right) \frac{1}{N}\sum_{i=1}^N \gamma_i[x_i(k_0) + v_i(k_0)].\end{aligned}$$

Thus, $b_i = \frac{\left(\sum_{j=1}^N b_j\gamma_j\right)}{N}\gamma_i, i = 1, 2, \cdots, N$. Denote $s_i = Nf^*b_i$ ($i = 1, 2, \cdots, N$) and assume $S = \mathrm{diag}\,(s_1, s_2, \cdots, s_N)$. Then SAS has all nonnegative elements and $\mathbf{1}^{\mathrm{T}}S\mathcal{L} = Nf^*C_b^{\mathrm{T}}\mathcal{L} = \mathbf{0}$. By Lemma 2.2, $\mathcal{G}$ is weight balanced, i.e., (H_5) holds. ∎

Remark 3.6 In conventional consensus work[21], (H_1), (H_3)–(H_5) are proved to be sufficient conditions for ensuring mean square average consensus. Here, from Theorem 3.5, one can see that they are also necessary to achieve mean square bipartite average consensus.

4 Simulation

In this section, we present two simulation examples to illustrate our theoretical results.

Example 4.1 Consider a group of 5 agents where each agent is described by (1). The communication interactions among 5 agents are represented by a signed digraph $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1, \mathcal{A}_1)$ as illustrated in Figure 1, where $\mathcal{V} = \{1, 2, 3, 4, 5\}$. Obviously, agent 5 is the root of $\mathcal{G}_1$, and hence, it has a spanning tree. From Figure 1 we know that $a_{21} = 1$, $a_{15} = -2$, $a_{34} = -1$, and $a_{45} = 1$. By definition, $\mathcal{G}_1$ is structurally balanced. Assume that $\mathcal{L}_1$ is the Laplacian of $\mathcal{G}_1$. Then, the eigenvalues of $\mathcal{L}_1$ are 1, 2, 1, 1, 0 and left and right eigenvectors associated with eigenvalue 0 are $C_b = (b_1, b_2, \cdots, b_5)^{\mathrm{T}} = (0\ 0\ 0\ 0\ {-1})^{\mathrm{T}}$ and $C_r = (1\ 1\ 1\ {-1}\ {-1})^{\mathrm{T}}$, respectively.

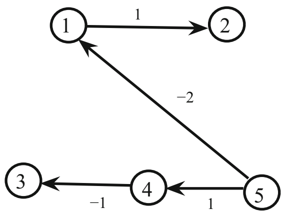

Figure 1 Signed digraph $\mathcal{G}_1$

Considering the effects of measurement noise, we choose $h(k) = \frac{1}{k+1}, k = 0, 1, \cdots$, in protocol (2). It is known that $h(k)$ satisfies (H_3) and (H_4). Suppose that the initial states are given by $X_1(0) = (-4 \quad 2)^{\mathrm{T}}$, $X_2(0) = (5 \quad 1)^{\mathrm{T}}$, $X_3(0) = (0 \quad 1)^{\mathrm{T}}$, $X_4(0) = (2 \quad -2)^{\mathrm{T}}$, and $X_5(0) = (-2 \quad 4)^{\mathrm{T}}$. Let $\mathcal{V}_{11} = \{1, 2, 3\}$ and $\mathcal{V}_{12} = \{4, 5\}$. The positions and velocities of agents in $\mathcal{V}_{11}$ and $\mathcal{V}_{12}$ evolve by applying the protocol in (2), which are presented in Figures 2 and 3, respectively. From Theorem 3.1 and Definition 2.3 we know that $x_i(t)/v_i(t)$ $(i = 1, 2, \cdots, 5)$ will converge to $\pm\xi_x^*/\pm\xi_v^*$ in mean square sense. Moreover, $E(\xi_x^*) = \sum_{j=1}^{5} b_j[x_j(0) + v_j(0)] = -2$ and $E(\xi_v^*) = 0$. They are validated by Figures 2 and 3, respectively.

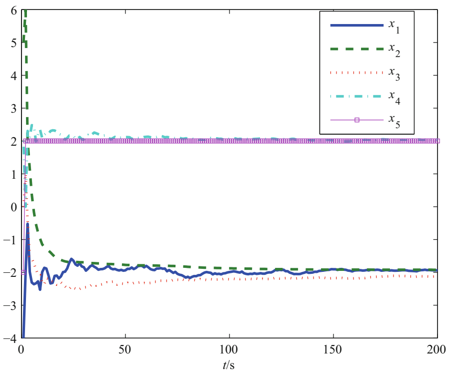

Figure 2 Position curves of agents over $\mathcal{G}_1$

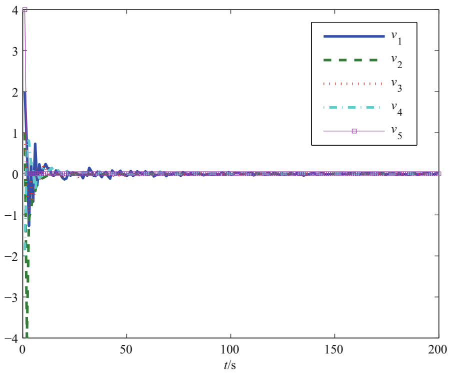

Figure 3 Velocity curves of agents over $\mathcal{G}_1$

Example 4.2 Particularly, suppose the communication interactions among the five agents in Example 4.1 are represented by signed digraph $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2, \mathcal{A}_2)$, as illustrated in Figure 4, where $\mathcal{E}_2 = \{(1, 2), (2, 5), (5, 1), (5, 4), (4, 3), (3, 1)\}$ and $a_{13} = 1$, $a_{15} = -2$, $a_{21} = 3$, $a_{34} = -1$, $a_{52} = -3$ and $a_{54} = 1$. Obviously, $\mathcal{G}_2$ not only satisfies (H_1) and (H_2), but also is weight balanced. The eigenvalues of Laplacian $\mathcal{L}_2$ are $4.6040 \pm 2.2479\jmath$, $0.3960 \pm 0.5480\jmath$ ($\jmath^2 = -1$), 0 and right eigenvector associated with eigenvalue 0 is $C_r = (-1, -1, -1, 1, 1)^{\mathrm{T}}$.

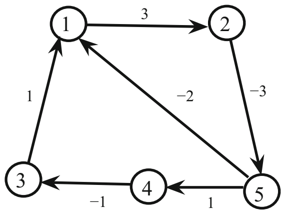

Figure 4 Signed digraph $\mathcal{G}_2$

To reduce the detrimental effects of measurement noise, $h(k) = \frac{k}{k_2+1}, k = 0, 1, \cdots$ is chosen in Protocol (2). Clearly, $h(k)$ satisfies (H_3) and (H_4). The initial states are $X_1(0) = (-4\ 2)^{\mathrm{T}}$, $X_2(0) = (5\ 1)^{\mathrm{T}}$, $X_3(0) = (2\ 0)^{\mathrm{T}}$, $X_4(0) = (1\ \ -2)^{\mathrm{T}}$, $X_5(0) = (-2\ 4)^{\mathrm{T}}$.

The positions and velocities of the five agents evolve by applying the protocol in (2), which are illustrated in Figures 5 and 6, respectively. From Theorem 3.5 we know that the position and velocity of each agent will converge in mean square to $\pm\xi_x^*$ and $\pm\xi_v^*$, respectively. Moreover, $E(\xi_x^*) = \frac{1}{5}\sum_{j=1}^{5}\gamma_j[x_j(0) + v_j(0)] = -1$ and $E(\xi_v^*) = 0$. They are validated by Figures 5 and 6, respectively.

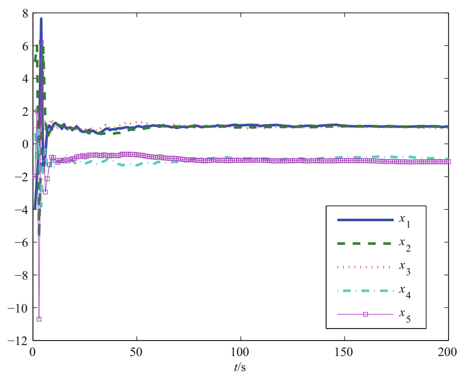

Figure 5 Position curves over $\mathcal{G}_2$

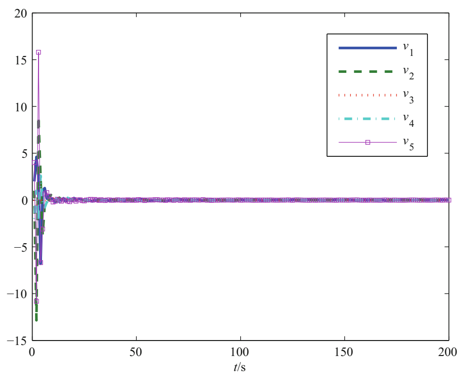

Figure 6 Velocity curves over $\mathcal{G}_2$

5 Conclusion

The bipartite consensus problem is investigated for discrete-time double-integrator MASs in the presence of measurement noise. Necessary and sufficient bipartite consensus conditions are established, where the signed graph is required to be structurally balanced and have a spanning tree. Further, if the signed digraph is weight balanced, mean square bipartite average consensus is achieved. Within this theoretical framework the switching communication topology will be our focus in the future work.

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